Journal of Statistical Physics, Vol. 70, Nos. 5/6, 1993

The Motion of a Tagged Particle and Nonhomogeneous Media in R^1

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Received April 3, 1990; final September 4, 1992

We analyze the asymptotic behavior of a tagged particle inside an infinite system of identical elastic point masses. The main objective is to study "very nonhomogeneous media"—particles which are more and more dispersed far from the origin. We suggest that the limit motion of a tagged particle may serve to classify media in the nonhomogeneous case as well as in the homogeneous case.

KEY WORDS: Tagged particle motion; scaling limit; nonhomogeneous media.

INTRODUCTION

The motion of a tagged particle in a chaotic bath forms a central problem in statistical physics. A rigorous mathematical study of the problem can illuminate some points of the classical dynamics of gases or liquids.

It had been widely accepted that the system after a while loses its memory, but the theoretical or numerical study of the motion of a tagged particle shows that its velocity correlation follows a pattern similar to that of problems of nonequilibrium statistical mechanics,[1] i.e., slow decay, and so the approximation of the system by one without memory is rather crude.

There have been extensive studies of the motion of a tagged particle. See Dürr *et al.*,[3,4] Szász and Tóth,[13,14] and Sinai and Soloveichik[11] for models, theorems, and discussions of open problems with regard to equilibrium Gibbs states.

Recently, there has been increasing interest in media out of equilibrium, e.g., in media distorted under shear. The rigorous mathematical

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study of the motion of a tagged particle due to particle interaction might show how far from equilibrium the medium is. From that point of view it seems natural to classify media by saying that two of them are equivalent if in both cases the stochastic process that represents the limit motion of a particle under scaling is the same.

At the moment it seems premature to evaluate the quality of such a classification; one needs more thorough mathematical results, and a relation of these results to physical phenomena, but it is noteworthy that this kind of classification is at least based on an idea similar to that of some experimental techniques.

In this paper, we will only deal with positions of particles in a gas of identical elastic masses. We set the additional tagged particle with the same mass at a specific site in the medium.

In the one-dimensional case under study we will analyze relatively simple situations, but even there, some unexpected phenomena appear. We discuss this in Section 3.

To analyze the limit behavior, we introduce a renormalization. We call the standard renormalization the $\sqrt{t}$ renormalization. The name standard renormalization is derived from the homogeneous case, where under $\sqrt{t}$ renormalization, there exists the scaled limit motion of a colliding tagged particle for a variety of models.[8] Recall also the theorem by Donsker for random walks. If the particles are more and more dispersed, then the standard $\sqrt{t}$ renormalization is not sufficient, in the sense that under this scaling the particle will still "escape to infinity" as in the case of free motion.

We will study mostly the class $[0]^s$ of media (see Definition 1 in Section 1), where the standard scaling is not sufficient in the above sense.

We consider a collection of particles (identical point masses) with positions x_k, $k \in Z$ ($k \in N$ if we assume initially only positive positions). We suppose that $x_k \leqslant x_{k+1}$. This ordering is always possible if, with probability one, $\{x_k\}$ has no point of accumulation. Now we endow each particle x_k with a random constant velocity v_k. We suppose that $\{x_k\}$ and $\{v_k\}$ are independent systems of random variables, and that the v_k are identically distributed independent random variables.

We define $x_k(t) = x_k + v_k t$, and assume that, if $i \neq j$, then $\{t: x_i(t) = x_j(t)\}$ does not contain a nondegenerate interval. This requirement is fulfilled if, for example, the v_k are of continuous type, or if $x_k \neq x_j$ for $k \neq j$. The particle $x_0 = 0$ is considered the tagged one and we assume that, at time zero, there is no other particle at the origin.

If we only consider the particles in R^+, then we add particles in R^-, symmetrically distributed with respect to the origin and such that $\{x_k\}_{k=1}^{\infty}$ and $\{x_k\}_{k=-\infty}^{-1}$ are independent systems.

Now we define the trajectory of the tagged particle under elastic collisions by the deterministic Harris theorem,[6,12]

$$y_0(t) = \lim_{n \to \infty} \operatorname{med}(x_{-n}(t), x_{-n+1}(t), \dots, x_0(t), \dots, x_n(t))$$

In our case, in order to satisfy the conditions of this theorem, it is only required that

$$\lim_{k \to +\infty} x_k = +\infty, \qquad \lim_{k \to +\infty} x_k + v_k t = +\infty$$

$$\lim_{k \to -\infty} x_k = -\infty, \qquad \lim_{k \to -\infty} x_k + v_k t = -\infty$$

almost everywhere.

If, for example, $x_{|k|} = |k|^{1/2}$, $k \in Z$, then we have to ask for the existence of $E[v_k^2]$.

We always assume that $E|v_k| = 1$, $P(v_k = 0) = 0$, and that v_k has a symmetric distribution. The last assumption is made to avoid some technical difficulties. The second one clearly is needed to avoid barriers.

We look for the limit behavior of $(h(A))^{-1} y_0(At)$, where $h(A)\colon R^+ \to R^+$, and make use of the fundamental lemma by Harris,[6]

$$\{y_0(At) < \alpha\} \equiv \{Z_A > 0\}$$

where

$$Z_A = Z_A(\alpha, t) = \chi[x_0 + v_0 At < \alpha] + \sum_{k=1}^{\infty} \chi[x_k + v_k At < \alpha]$$
$$- \sum_{k=-\infty}^{-1} \chi[x_k + v_k At \geqslant \alpha]$$

Here and throughout the paper $\chi[\cdot]$ is the indicator function of $[\cdot]$.

In this lemma we can include a renormalization factor $h(A) > 0$, and consider $y_0(At) < \alpha h(A)$. It is always required that $h(A) = o(A)$.

Using the lemma, we will deal with sums of random variables, and the limit behavior of these sums will serve for the classification of media.

1. STATEMENT OF RESULTS

If the medium is formed by only one (the tagged) particle $x_0 = 0$, then

$$y_0(t) = x_0(t) = v_0 t$$

Therefore

$$t^{-\alpha}|y_0(t)| \to +\infty \quad \text{a.e.} \quad \text{for any} \quad \alpha < 1$$

By the symmetry of v_k, and using, for example, the standard scaling, we get

$$P(t^{-1/2}y_0(t) < \alpha) \to 1/2$$

We will see later that the same will be true for particles of the medium which are more and more dispersed far from the origin. So we start with the following:

Definition 1. We say that $\{x_k\} \in [0]^s$ if and only if

$$\forall_{\alpha \in R} P(t^{-1/2}y_0(t) < \alpha) \to 1/2 \qquad \text{if} \quad t \to \infty$$

("s" means the standard renormalization).

The $t^{1/2}$ scaling might be substituted by any function h: $\mathbb{R}^+ \to \mathbb{R}^+$, $h(t) = o(t)$ and we introduce the class $[0]^h$.

Definition 2. We say that $\{x_k\} \in [0]^h$, where $h: R^+ \to R^+$, if

$$\forall_{\alpha \in R} P[(h(t))^{-1} [y_0(t)] < \alpha] \to 1/2 \qquad \text{if} \quad t \to \infty$$

It is clear that if

$$\{x_k\} \in [0]^{h_1} \quad \text{and} \quad h_2 < h_1, \qquad \text{then} \quad \{x_k\} \in [0]^{h_2}$$

In the symmetric case, this condition is equivalent to

$$\forall_{\alpha \in R} P[(h(t))^{-1} |y_0(t)| < \alpha] \to 0$$

By the term symmetric case, we mean that $x_k \stackrel{D}{=} -x_{-k}$, $k \in N$, and $\{x_k\}_1^{\infty}$ and $\{x_k\}_{-1}^{-\infty}$ are independent systems of random variables. In this paper we will deal with symmetric or close to symmetric positions. In view of Harris' lemma (cf. Introduction), the first condition is easier to analyze.

Theorem 1. In the symmetric case, if $\{x_k\} \in [0]^h$ and $\{\tilde{x}_k\} \in [0]^h$, where $\{x_k\}$ and $\{\tilde{x}_k\}$ are independent systems of independent random variables, then

$$\{x_k\} \cup \{\tilde{x}_k\} \in [0]^h$$

(because the tagged particle is set after the union, it is counted only once).

In the following theorem we will investigate the class $[0]^s$ in the case of deterministic (symmetric) positions.

Theorem 2. Let $x_k = F(k) > 0$, $k \geqslant 0$, and $x_{-k} = -F(k)$. The function $F(x)$ is increasing, and moreover has an increasing first derivative $F'(x)$. [Originally we define only $F(k)$, and $F(x)$ is an extension of $F(k)$. Clearly we can make $F \in C^1$.] Assume also that v_k is separated from zero, or has bounded density.

Then

$$\{x_k\} \in [0]^s$$

We shall discuss these very strong conditions imposed on the function F, which defines positions x_k, later. On the other hand, since we are looking for the classification of positions of particles, any condition on v_k is of minor importance.

In this theorem we have analyzed the case of deterministic positions.

We will now introduce the concept of asymptotic bounds for a random medium, which will give us a tool to deal with more interesting (random positions) situations.

Let $\{x_k^{(i)}\}$, $i = 1, 2, 3$, be three random media such that

$$\sum_{k=-\infty}^{\infty} P(x_k^{(2)} > x_k^{(3)}) < \infty, \qquad \sum_{k=-\infty}^{+\infty} P(x_k^{(2)} < x_k^{(1)}) < \infty$$

and for any integer k, $x_k^{(1)} < x_k^{(3)}$.

Consider the given scaling $h(A)$: $R^+ \to R^+$. By Harris' lemma from the Introduction we will study the limit behavior of $Z_A^{(i)} = Z_A^{(i)}(\alpha h(A), t)$ for media $\{x_k^{(i)}\}$, $i = 1, 2, 3$. We have

$$Z_A^{(i)}(\alpha h(A), t) = \chi[v_0 At < \alpha h(A)] + \sum_{k=1}^{+\infty} \chi[x_k^{(i)} + v_k At < \alpha h(A)]$$
$$- \sum_{k=-\infty}^{-1} \chi[x_k^{(i)} + v_k At \geqslant \alpha h(A)]$$

Suppose that for $i = 1, 3$, $Z_A^{(i)}$ needs the same h^* to become in the limit a random variable with no atom at zero. (We stress the difference between h and h^*; h is fixed scaling factor, and h^* a suitable factor of secondary importance to the study of Z.)

More precisely, for $i = 1, 3$, and each $n > 0$,

$$\left(\frac{Z_A^{(i)}(\alpha_k h(A), t_k)}{h^*(A)}, k = 1, 2, ..., n \right)$$
$$\xrightarrow{D} [W(\alpha_k, t_k), k = 1, ..., n]$$

$W(\alpha_k, t_k)$ (multidimensional r.v.) has no atom at zero for any choice of (α_k, t_k).

We ask for the convergence of joint distributions but we do not ask for any consistency conditions for joint distribution laws of $W(\alpha_k, t_k)$, $k = 1,..., n$, the same for $i = 1, 3$.

In the case described above we say that media $\{x_k^{(1)}\}$ and $\{x_k^{(3)}\}$ form asymptotic bounds for a medium $\{x_k^{(2)}\}$.

Theorem 3. If $\{x_k^{(1)}\}$ and $\{x_k^{(3)}\}$ form asymptotic bounds for $\{x_k^{(2)}\}$, then $\lim_{A \to \infty} (h(A))^{-1} y_0^{(i)}(At)$ does not depend on $i = 1, 2, 3$ (in the sense of convergence of finite-dimensional distributions).

Corollary. If $W(\alpha, 1)$ is Gaussian with mean zero for any α, then

$$\{x_k^{(2)}\} \in [0]^h$$

In this case

$$P[Z_A^{(i)}(\alpha h(A), 1) > 0] \to 1/2$$

and by Harris' lemma

$$P(h(t)^{-1} (y_0^{(i)}(t)) < \alpha) \to 1/2$$

With the help of Theorem 3, we can prove that more media that were obtained from Theorem 1 belong to a class $[0]^h$.

Now we will give an application of Theorem 3 for the case when the differences $x_k - x_{k-1}$ are independent random variables.

If $\xi_k = x_k - x_{k-1}$, $k \in Z$, are independent identically distributed random variables, then $\{x_k\}$ is a renewal process and has been analyzed by Major and Szász.[8] We will discuss this model in Section 3. In view of our goal, it is natural to study the case of the ξ_k independent, with increasing mean. Here we seek the common form of distribution, but with different parameters. We examine only the example of the exponential distribution for ξ_k. We define ξ_k for $k > 0$, and we assume symmetry, i.e.,

$$\xi_{-k} \overset{D}{=} -\xi_k, \qquad k \in N$$

(We assume independence of $\{x_k\}_{-\infty}^{-1}$ with respect to $\{x_k\}_1^{\infty}$.) Then we have the following result.

Theorem 4. Let $E(\xi_1) = 1$. Now, $E(\xi_k) = k^\alpha - (k-1)^\alpha$, $k > 0$, $\alpha > 1$, and ξ_k ($k \in N$) have the exponential distribution. Then

$$\{x_k\} \in [0]^s$$

In the study of, for example, more and more dispersed particles far from the origin, in the proper sense $\{x_k\} \in [0]^s$, $\sqrt{t}$ renormalization was "not enough." It is natural to ask if there exists a scaling such that the limit motion of the tagged particle becames a stochastic process (not infinity) with no atom at zero, and call this scaling "natural renormalization."

The natural renormalization does not always exist.

We will see in Theorem 4 that if $x_k = k^\alpha$ and $\alpha \to \infty$, then the natural renormalizing factor will be t^β with $\beta \to 1$.

So, for positions, e.g., $x_k = e^k$, the natural renormalizing factor should be at least t. But, on the other hand, it must be $o(t)$.

We can also look for natural renormalization for media which are denser far from the origin, as in the following example:

$$x_k = \sqrt{k}, \quad F(x) = \sqrt{x}, \quad f(x) = F^{-1}(x) = x^2, \quad \text{where} \quad k, x > 0$$

and we assume symmetry. Then, without any renormalizing factor, we obtain the positive limit for the variance of $y_0(At)$. If, for example, $v_k = \pm 1$, then

$$P(y_0(A) < 1) = P(Z_A(1, 1) > 0) = P\left(\frac{Z_A(1, 1)}{A} > 0\right)$$

We have

$$A^{-1} E(Z_A(1, 1)) \sim \frac{1}{2} \frac{(A+1)^2 - (A-1)^2}{A} \to 2$$

$$A^{-2} \operatorname{Var}(Z_A(1, 1)) \sim \frac{1}{4} \frac{(A+1)^2 + (A-1)^2}{A^2} \to \frac{1}{2}$$

and we apply the standard central limit theorem.

We will frequently use $f(t) = F^{-1}(t)$. The function $f(t)$ plays a very special role in the proofs. This should be clear if we note that $f(t)$ represents approximately the number of particles $\{x_k\}$ in the interval $[0, t]$ at time zero, and using Harris' lemma from the Introduction, this number of particles is crucial.

In the general case of deterministic positions, $x_k = F(k)$, $f(t) = F^{-1}(t)$, where F is an increasing function; then if we set $h(t) = [f(t)]/f'(t)^{1/2}$, we can expect under easy but informal calculations that for each $\alpha > 0$ and $t > 0$

$$f(A)^{-1/2} E(Z_A(\alpha h(A), t)) \to c_1(\alpha, t) > 0$$

$$f(A)^{-1} \operatorname{Var}(Z_A(\alpha h(A), t) \to c_2(\alpha, t) > 0$$

and $h(t)$ could be a natural renormalization. If $F(t)=e^t$, $h(t)=t(\ln t)^{1/2}$, which is inadmissible.

The case $x_k=k^{1/\eta}$ will be treated in Theorem 5, leaving more complex problems for another study.

Theorem 5. Let $f(x)=F^{-1}(x)=x^\eta$, $\eta>0$. If $|v_k|=1$, then the limit process of $y_0(At)/A^{1-\eta/2}$ is the Gaussian process with mean zero and covariance

$$E(y(s)\cdot y(u))=\frac{1}{2\eta^2}\frac{1}{s^{\eta-1}u^{\eta-1}}[\min(s,u)]^\eta, \qquad s,u>0$$

As before, "limit" refers to the convergence of the finite-dimensional distributions. It is not difficult to prove tightness for $\eta<2$. If $\eta\geqslant 2$, then $y_0(At)/A^{1-\eta/2}$ is not tight. (The limit process does not have continuous paths.) The assumption $|v_k|=1$ is only a simplifying one, and we might easily extend the result to arbitrary velocities (with different limiting covariances). In this case the proof of tightness, if this is present, seems to be difficult, mostly because we do not have any stationarity properties of the process $y_0(At)$. This process does not seem to be close to some Markov one either.

2. PROOFS OF THE THEOREMS

Proof of Theorem 1. Suppose $\alpha>0$. We will use the following partition of $Z_t(\alpha h(t),1)$ (instead of A we put t) for x_k independent random variables, $\{x_k\}\in[0]^h$, $Z_t(\alpha h(t),1)=\xi_t+\varepsilon_t$, where ξ_t is the symmetric part,

$$\xi_t=\sum_{k=1}^{\infty}\chi[x_k+v_kt\leqslant -\alpha h(t)]-\sum_{k=-\infty}^{-1}\chi[x_k+v_kt\geqslant \alpha h(t)]$$

By symmetry, $P(\xi_t>0)\to 1/2$ if $P(\xi_t=0)\to 0$.

But this is fulfilled in the case of an infinite number of particles on the right of the origin (then on the left as well). In this case

$$P(-\varepsilon_t<\xi_t<\varepsilon_t)\to 0$$

and for any $A\in\mathbb{R}$

$$P(A-\varepsilon_t<\xi_t<A+\varepsilon_t)\to 0$$

and moreover uniformly with respect to A. Therefore the proof of the theorem is reduced to standard probabilistic calculus.

If both media are finite, there is almost nothing to do and, if exactly one, say $\{x_k\}$, is infinite, then, for this medium $P(Z_t(\alpha h(t), 1) = k) \to 0$, and the theorem results in a straightforward way.

Proof of Theorem 2. For simplicity, we will give the proof in the case of bounded density $g(v)$ for v_k. We will show that

$$P(Z_t(\alpha \sqrt{t}, 1) > 0) \to 1/2$$

Without loss of generality we may set $\alpha = 1$. Let $Z(t) = Z_t(\sqrt{t}, 1)$ and $f(t) = F^{-1}(t)$.

The suitable factor is $h^*(t) = (f(t))^{-1/2}$, so that we consider $P(Z(t) > 0) = P(f(t)^{-1/2} Z(t) > 0)$. Since Z_t is a sum of independent random variables, by the central limit theorem, it suffices to show that

$$f(t)^{-1/2} E(Z(t)) \to 0, \qquad \liminf f(t)^{-1} \operatorname{Var}(Z(t)) > 0 \qquad \text{if} \quad t \to \infty$$

Since $f(t) \to \infty$,

$$\begin{aligned}
&(f(t))^{-1/2} E(Z(t)) \\
&\quad \sim (f(t))^{-1/2} \left\{ \sum_{k=1}^{\infty} P(x_k + vt < \sqrt{t}) - \sum_{k=1}^{\infty} P(x_k + vt > \sqrt{t}) \right\} \\
&\quad = (f(t))^{-1/2} \left\{ \sum_{k=1}^{\infty} P[f^+(vt + \sqrt{t}) > k] - \sum_{k=1}^{\infty} P[f^+(vt - \sqrt{t}) > k] \right\}
\end{aligned}$$

Here $f^+(x) = f(x)$ if $x > 0$, and is zero if $x \leqslant 0$. Hence,

$$(f(t))^{-1/2} E(Z(t)) \sim E[f^+(vt + \sqrt{t}) - f^+(vt - \sqrt{t})](f(t))^{-1/2}$$

As a matter of fact, in the first term we may suppose that $v > 0$ because if we set $A(t) = \{v \colon v < 0, vt + \sqrt{t} > 0\}$, then

$$\int_{A(t)} \frac{f(vt + \sqrt{t})}{[f(t)]^{1/2}}\, g(v)\, dv \leqslant c \frac{1}{\sqrt{t}} \frac{f(2\sqrt{t})}{[f(t)]^{1/2}} \to 0$$

In the same way we may suppose that $vt - \sqrt{t} > \sqrt{t}$, for these v:

$$f(tv + \sqrt{t}) - f(tv - \sqrt{t}) < 2[f(tv + 2\sqrt{t}) - f(tv)]$$

Now

$$(f(t))^{-1/2} [f(tv + 2\sqrt{t}) - f(tv)] \leqslant c f'(tv) \cdot \sqrt{t}\, (f(t))^{-1/2}$$

Now it is an easy task to prove that

$$\int_{1/\sqrt{t}}^{\infty} \frac{f'(tv)\sqrt{t}}{[f(t)]^{1/2}}\, g(v)\, dv \to 0$$

On the other hand,

$$\begin{aligned}(f(t))^{-1} E\{Z^2(t)\} &\geqslant \int_0^\infty \frac{f(tv)}{f(t)}\, g(v)\, dv \\ &\geqslant \int_1^\infty \frac{f(tv)}{f(t)}\, g(v)\, dv \\ &\geqslant P(v \geqslant 1) > 0\end{aligned}$$

This ends the proof of Theorem 2.

Proof of Theorem 3. For simplicity set $n=1$. Consider the set $\tilde{S}$ of integers k, for which

$$x_k^{(1)} \leqslant x_k^{(2)} \leqslant x_k^{(3)}$$

Now we construct $\tilde{Z}_A^{(i)}$,

$$\begin{aligned}\tilde{Z}_A^{(i)}(\alpha h(A), t) = \chi[v_0 At < \alpha h(A)] &+ \sum_{k>0, k\in\tilde{S}} \chi[x_k^{(i)} + v_k At < \alpha h(A)] \\ &- \sum_{k<0,\, k\in\tilde{S}} \chi[x_k^{(i)} + v_k At \geqslant \alpha h(A)]\end{aligned}$$

Construct also $\tilde{y}_0^{(i)}(t)$ (particles $x_k^{(i)}$ with $k \notin \tilde{S}$ disappear). By the assumptions of the theorem we have for $i = 1, 2, 3$

$$\frac{Z_A^{(i)}(\alpha h(A), t) - \tilde{Z}_A^{(i)}(\alpha h(A), t)}{h^*(A)} \to 0 \qquad \text{a.e.}$$

Recall that $h^*(A) \to \infty$, and $\#\{k: k \notin \tilde{S}\}$ is finite with probability one

Now, for $i = 1, 3$,

$$\begin{aligned}\lim_{A\to\infty} P\left(\frac{y_0^{(i)}(At)}{h(A)} < \alpha\right) &= \lim_{A\to\infty} P(Z_A^{(i)}(\alpha h(A), t) > 0) \\ &= \lim_{A\to\infty} P\left(\frac{Z_A^{(i)}(\alpha h(A), t)}{h^*(A)} > 0\right) \\ &= \lim_{A\to\infty} P\left(\frac{\tilde{Z}_A^{(i)}(\alpha h(A), t)}{h^*(A)} > 0\right) \\ &= \lim_{A\to\infty} P\left(\frac{\tilde{y}_0^{(i)}(At)}{h(A)} < \alpha\right)\end{aligned}$$

The third equality holds, because $W(\alpha, t)$ has no atom at zero. By assumption, every limit probability in the sequel does not depend on $i = 1, 3$.

But $\tilde{Z}_A^{(3)} \geqslant \tilde{Z}_A^{(2)} \geqslant \tilde{Z}_A^{(1)}$ and moreover $\tilde{y}_0^{(1)} \leqslant \tilde{y}_0^{(2)} \leqslant \tilde{y}_0^{(3)}$. Therefore

$$\lim_{A \to \infty} P\left(\frac{\tilde{y}_0^{(i)}(At)}{h(A)} < \alpha\right)$$

does not depend on $i = 1, 2, 3$ and going back we see that the same is true without the tilde.

Proof of Theorem 4. Note that $E(x_k) = k^\alpha$, $\operatorname{Var}(x_k) = ck^{2\alpha - 1} + o(k^{2\alpha - 1})$. Let ε be a "small positive number." We have

$$P(x_k < (k - k^{1 - 1/(2\alpha) - \varepsilon})^\alpha) = P\left(\frac{\xi_1 + \cdots + \xi_k - k^\alpha}{k^{\alpha - 1/2}} < \frac{(k - k^{1 - 1/(2\alpha) - \varepsilon})^\alpha - k^\alpha}{k^{\alpha - 1/2}}\right)$$

Now,

$$\frac{k^\alpha - (k - k^{1 - 1/(2\alpha) - \varepsilon})^\alpha}{k^{\alpha - 1/2}} \sim k^\beta$$

where $\beta = 1/2 - 1/(2\alpha) - \varepsilon^* > 0$, and ε^* is another small positive number.

If we deal with a normal random variable η, then

$$\sum_{k=1}^{\infty} P(\eta > k^\beta) \quad \text{converges}, \qquad \text{where} \quad \beta > 0$$

so we will estimate the difference between the standard normal distribution and the standardized sum $S_k = \xi_1 + \cdots + \xi_k$.

Using a classical estimation for moments of $S_k = x_k$, we can easily see that $E\{|[x_k - E(x_k)]/\sigma_k|^\rho\}$ is uniformly bounded in k for any fixed ρ. Here $\sigma_k = (\operatorname{Var} x_k)^{1/2}$ (cf. Petrov,[9] p. 60). Therefore, setting $G(x)$, the distribution function of $[x_k - E(x_k)]/\sigma_k$, and using a standard calculation, we get

$$|G_k(x) - \Phi(x)| < \frac{c_P}{1 + x^P} \qquad \text{for any} \quad p$$

(see Petrov,[9] Theorem 4.9).

Choosing p big enough, we obtain $(1/2 - 1/2\alpha)\, p > 1$, whence, for any $\alpha > 1$, the series

$$\sum_{k=1}^{\infty} P[\xi_1 + \cdots + \xi_n \lesseqgtr (k \mp k^{1 - 1/(2\alpha) - \varepsilon^*})^\alpha]$$

converges.

For $\alpha > 1$ we can now apply Theorem 3 for $x_k^{(2)} = x_k$,

$$x_k^{(1)} = \begin{cases} -(|k| + |k|^{1-1/(2\alpha)-\varepsilon^*}) & \text{if} \quad k<0 \\ (k - k^{1-1/(2\alpha)-\varepsilon^*}) & \text{if} \quad k>0 \end{cases}$$

and

$$x_k^{(3)} = \begin{cases} -(|k| - |k|^{1-1/(2\alpha)-\varepsilon^*}) & \text{if} \quad k<0 \\ (k + k^{1-1/(2\alpha)-\varepsilon^*}) & \text{if} \quad k>0 \end{cases}$$

The proof that $\{x_k^{(i)}\} \in [0]^s$, $i = 1, 3$, is similar to that of Theorem 2 and we can apply the corollary to Theorem 3.

Proof of Theorem 5. First note that

$$\lim_{A \to \infty} A^{-\eta/2} E(Z_A(\alpha^*, s)) = \eta \alpha s^{\eta - 1}$$

and

$$\lim_{A \to \infty} A^{-\eta} \operatorname{Var}(Z_A(\alpha^*, s)) = \tfrac{1}{2} s^\eta$$

Here $\alpha^* = \alpha A^{1-\eta/2}$ and the suitable factor is $A^{\eta/2}$.

Set $Z_A^*(\alpha, s) = A^{-\eta/2} Z_A(\alpha^*, s)$. Now

$$\begin{aligned} &P(Z_A(\alpha^*, s) > 0, Z_A(\beta^*, u) > 0) \\ &\quad = P\left(\frac{Z_A^*(\alpha, s) - \eta\alpha s^{\eta-1}}{\eta s^{\eta-1}} \leqslant \alpha, \frac{Z_A^*(\beta, u) - \eta\beta u^{\eta-1}}{\eta u^{\eta-1}} \leqslant \beta\right) \end{aligned}$$

Now it is an easy task to show that

$$\lim_{A \to \infty} \operatorname{cov}(Z_A^*(\alpha, s), Z_A^*(\beta, u)) = \tfrac{1}{2}[\min(s, u)]^\eta$$

and hence Theorem 5 is proved.

3. DISCUSSION

In this first attempt to solve the "inverse problem" starting from the scaling limit of the motion of tagged particle and ending with the probability law of the chaotic bath, we will present some discussion of our results and state some open problems.

Problem 1. Generalize Theorem 1 without the symmetry assumption or without the assumption of independence of $\{x_l\}$, $\{\tilde{x}_l\}$. Clearly, for

x_k, $k \in N$, not necessarily independent, we can put $x_{-k} = -x_k$, and if for almost all ω, $\{x_n(\omega)\} \in [0]^h$ and $\{\tilde{x}_n(\omega)\} \in [0]^h$, $n \in Z$, then, for these ω,

$$\{x_n(\omega)\} \cup \{\tilde{x}_n(\omega)\} \in [0]^h$$

Therefore $\{x_k\} \cup \{\tilde{x}_k\} \in [0]^h$.

Obviously the hypotheses of Theorem 2 seem to be very restrictive. One might think that if $\{x_k\} \in [0]^s$ and $|\tilde{x}_k| > |x_k|$, $x_{-k} = -x_k$, $\tilde{x}_{-k} = -\tilde{x}_k$, then $\{\tilde{x}_k\} \in [0]^s$.

We can easily give a family of counterexamples for the case $v_k = \pm 1$. These counterexamples are constructed on the following basis. If particles are highly dispersed in mean but with "condensation points," then this groups of particles will force, from time to time, the tagged one to return close to the origin. This case might be illustrated as shown in Fig. 1. In this picture, $\phi(t) = (F^{-1}(t))^{1/2}$ is defined to be a piecewise linear increasing function whose graph lies below that of $t^{1/4}$ and has two alternating types of slope. The larger one is constant, and the segments with this slope are of constant length. The other slope (a decreasing step function) is adjusted to the slope of $t^{1/4}$.

Recall from Section 2 that $f(t) = \varphi^2(t)$ is approximately the number of particles in $[0, t]$. Consequently, we will have an increasing number of particles in groups of high density.

Formally,

$$\phi(t + \sqrt{t}) - \phi(t) \nrightarrow 0$$

Then

$$(f(t))^{-1/2} E[f(t\,|v| + \sqrt{t}) - f(t\,|v|)] > \phi(t + \sqrt{t}) - \phi(t) \nrightarrow 0$$

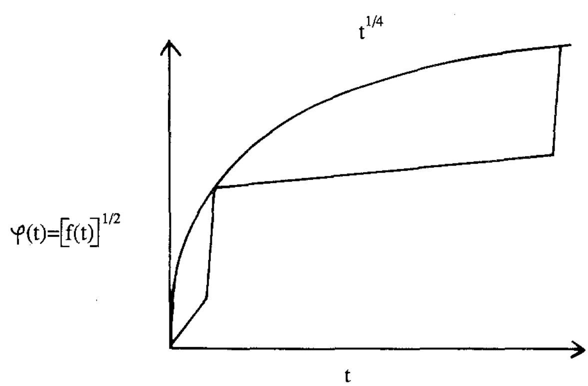

Fig. 1

but, on the other hand,

$$f(t)^{-1}\,[f(t+\sqrt{t})+f(t)]\to 2$$

Therefore $(f(t))^{-1}$ Var $Z(t)$ is bounded.

Now one easily obtains $\{x_k\}\notin[0]^s$.

Note. We could try to analyze the problem of Theorem 2 using the theory of regularly varying functions.[10] Suppose that $f(t)=F^{-1}(t)=t^\rho L(t)$, where $L(t)$ is a slowly varying function, $\rho<1$. Now, considering the "weaker version" of what is needed,

$$\int_0^\infty \frac{(tv+\sqrt{t})^\rho\,L(tv+\sqrt{t})-(tv)^\rho\,L(tv)}{t^{\rho/2}[L(t)]^{1/2}}\,P(dv)$$
$$=\int_0^\infty \frac{[(tv+\sqrt{t})^\rho-(tv)^\rho]\,L(tv)}{t^{\rho/2}[L(t)]^{1/2}}\,g(v)\,dv$$
$$+\int_0^\infty \frac{[L(tv+\sqrt{t})-L(tv)](tv+\sqrt{t})^\rho}{t^{\rho/2}[L(t)]^{1/2}}\,g(v)\,dv$$

The first integral tends to zero if $t\to\infty$ under a suitable assumption (cf. Seneta,[10] Theorems 2.6, 2.7), but, dealing with the second, one has to prove that

$$\frac{L(t+\sqrt{t})-L(t)}{t^{-\rho/2}}\to 0$$

This would lead to a special kind of very slowly varying function.[2]

Inside the class $[h]^N$ of random media with h being the natural renormalization for them, we may introduce the equivalence relation between two media $\{x_k\}$ and $\{y_k\}$. Let us say that $\{x_k\}\sim_h\{y_k\}$ if the stochastic processes that represent the scaled limit of the motion of the tagged particle are equivalent, i.e., have the same finite-dimensional distributions.

Now the following problem occurs:

Problem 2. If $\{x_k\}\in[h]^N$ and $\{y_k\}\in[0]^h$, then, is it true that $\{x_k\}\sim_h\{x_k\cup y_k\}$?

Major and Szász[8] have considered homogeneous media with $\sqrt{t}$ scaling.

In the case of renewal positions, i.e., $\xi_k=x_k-x_{k-1}$ being i.i.d.r.v. with mean μ and variance σ^2, they obtained that $y_0(At)/\sqrt{A}$ converges weakly to the non-Markovian Gaussian process with covariance function

$$\mu^{-1}E\,|v|\min(t,s)+\mu^{-3}(\sigma^2-\mu^2)\,\tfrac{1}{2}E\min(t\,|u|,s\,|v|)$$

with s, $t>0$, and u, v are i.i.d.r.v. with the same symmetric distribution law as the velocities v_k.

Comparing this result with that of Szatzschneider[15] for independent positions $x_k = \xi_k + k$, where the covariance function is

$$E\,|v|\,\min(t, s) - \tfrac{1}{2}E\min(t\,|u|, s\,|v|)$$

one gets that:

1. An "independent" medium after an ordering will never be a "renewal" one. This is only possible in the deterministic case. Moreover, "independent medium" is not equivalent to any "renewal medium."

2. Two renewal media are equivalent if and only if $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$. We can repeat here the proof of the non-Markovian character.[15]

3. It is possible to "mix independently" (i.e., consider $\{\tilde{x}_k\} \cup \{\tilde{\tilde{x}}_k\} = \{x_k\}$) a renewal process with an independent process and obtain Brownian motion in the limit. This is always possible if for the renewal process, $\sigma^2 > \mu^2$, choosing for the independent process $\hat{\mu} = E(x_k - x_{k-1})$ such that

$$\mu^{-3}(\sigma^2 - \mu^2) = (\hat{\mu})^{-1}$$

In this case the mixture is equivalent to a Poisson process.

Problem 3. Is it possible, using new techniques, to prove that for an infinitely divisible renewal process the limit process is Brownian motion? It suffices to show that it is a Markovian process. This would lead to the result that for these processes $(E(x_k - x_{k-1}))^2 = \mathrm{Var}(x_k - x_{k-1})$, a result related to the solution of an open problem: Is any infinitely divisible renewal process the Poisson process?[5]

We have proposed a new classification of media. We can mention the possibility of perhaps a "better classification" in the homogeneous case. It is easy to see that, if we start the moment of observation at fixed time A, then the limit results do not change. On the other hand, if $A \to \infty$ sufficiently fast, then the limit will be Brownian motion in many cases.[7] The rate o convergence of A which leads to Brownian motion would serve to classify media. This problem seems to be a difficult one.

ACKNOWLEDGMENTS

I would like to express my deep gratitude to the referees for their help in the preparation of the final version of the paper.

Thanks to them I made many substantial improvements to the text.

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